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Acceptance Angle of a Plasma Channel

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Acceptance Angle of a Plasma Channel

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Plasma channels are useful as guiding structures for high intensity electromagnetic radiation. A finite radius plasma channel ceases to act as a guiding structure if the f-number of the focusing used to couple radiation into the channel is too small. Expressions are derived for the critical f-number needed to achieve guiding in a finite radius plasma channel. This is done using both a ray tracing analysis, and an eigenmode expansion technique.

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I. INTRODUCTION

In order to guide ultra-intense ($> 10^{18}$ W/cm^2) laser pulses, plasma channels have to be employed. Such guiding is useful for applications such as laser wakefield accelerators [1], or x-ray lasers [2]. The guiding characteristics of plasma channels are well known [3–6]. When focusing the laser radiation into the channel, one would typically like to choose the lens such that the waist is located at the channel entrance, and such that the waist radius is matched to the channel [7]. There are cases, however, where this cannot be arranged, and the question arises as to whether, given a certain focusing configuration, there will any guiding at all. This report gives a formula for the smallest f-number lens that may be used such that guiding is achieved. It also extends the well known theory of spot size oscillations in a channel to the case where the channel radius is finite.

The outline of this report is as follows. The finite radius plasma channel and critical f-number are introduced in section II. A ray tracing analysis is employed to obtain the critical f-number in section III. An eigenmode expansion method is employed to obtain the critical f-number in section IV. An example is worked out in section V.

II. PLASMA CHANNEL

An ideal plasma channel has a radial density in the form

$$n(r) = n_0 \left(1 + \frac{r^2}{R_{ch}^2}\right) \tag{1}$$

where R_{ch} is the density doubling radius. Because the density increases in r without bound, the channel will guide radiation no matter what f-number is used to couple it into the channel. When the plasma channel is finite, however, there is a critical f-number below which the channel will not guide. In the following, we consider a density profile of the form

$$n(r) = \begin{cases} n_0 \left(1 + r^2/R_{ch}^2\right) & , \ r < R_m \\ 0 & , \ r \geq R_m \end{cases} \tag{2}$$

where R_m is the maximum channel radius. The index of refraction is therefore determined by

$$\eta^2 = \begin{cases} 1 - (k_p^2/k^2)\left(1 + r^2/R_{ch}^2\right) & , \ r < R_m \\ 1 & , \ r \geq R_m \end{cases} \tag{3}$$

where k_p is the plasma wavenumber evaluated at n_0 and k is the radiation wavenumber.

III. RAY TRACING ANALYSIS

The equations for the evolution of a ray in a material with a radially varying index of refraction are

$$\frac{d\theta}{dt} = c\frac{d\eta}{dr}\cos\theta \tag{4}$$

$$\frac{dr}{dt} = c\sin\theta \tag{5}$$

where θ is the angle between the ray trajectory and the z-axis, and we assumed $\eta \approx 1$. Substituting Eq. (3) for η gives

$$\frac{d\theta}{dt} = -c\frac{k_p^2}{k^2}\frac{r}{R_{ch}^2}\cos\theta \tag{6}$$

provided $r < R_m$. Taking $\theta \ll 1$, differentiating once, and substituting in Eq. (5) gives

$$\left(\frac{d^2}{dt^2} + \Omega^2\right)\theta = 0 \tag{7}$$

where

$$\Omega^2 = \frac{k_p^2}{k^2}\frac{c^2}{R_{ch}^2} \tag{8}$$

The solution to this equation is

$$\theta = \frac{\theta_0}{2}e^{i\Omega t} + c.c. \tag{9}$$

It should be noted that Ω is the frequency of oscillation of a ray, which is half the frequency of the spot size oscillation. Inserting the solution for θ into Eq. (5) and integrating gives

$$r = \frac{c\theta_0}{2i\Omega}e^{i\Omega t} + c.c. \tag{10}$$

If $r < R_m$ for all t we can consider the ray to be guided. That is, the condition for guiding is $r < R_m$, or

$$\theta_0\frac{k}{k_p}R_{ch} < R_m \tag{11}$$

Associating angle and f-number through $\theta = 1/2F$ gives the critical f-number as

$$F_c = \frac{1}{2}\frac{R_{ch}}{R_m}\frac{k}{k_p} \tag{12}$$

IV. EIGENMODE EXPANSION ANALYSIS

A source dependent expansion (SDE) [8] analysis for the case where $R_m \to \infty$ has been carried out in Ref. [9] where the equation for the spot size (neglecting the ponderomotive and relativistic terms) was given as

$$\frac{d^2 r_s}{dz^2} = \frac{4}{k^2 r_s^3} + \int_0^\infty d\chi (1-\chi) \left(1 + \frac{r_s^2}{2R_{ch}^2}\chi\right) e^{-\chi} \tag{13}$$

where $\chi = 2r^2/r_s^2$. To generalize this equation to the present case, we replace the upper limit of the integration with $2R_m^2/r_s^2$. Carrying out the integral gives

$$\frac{d^2 r_s}{dz^2} = \frac{4}{k^2 r_s^3} + f(r_s) \tag{14}$$

where

$$f(r_s) = \frac{k_p^2 r_s^2}{k^2 R_{ch}^2} \left[\frac{e^{-2R_m^2/r_s^2}}{r_s^5} g(r_s) - \frac{1}{r_s} \right] \tag{15}$$

and

$$g(r_s) = 4R_m^2 (R_{ch}^2 + R_m^2) + 2R_m^2 r_s^2 + r_s^4 \tag{16}$$

This can be put in the form

$$\frac{1}{2}\left(\frac{dr_s}{dz}\right)^2 + V(r_s) = C \tag{17}$$

where C is a constant and

$$V(r_s) = \frac{2}{k^2 r_s^2} - \int^{r_s} dr_s' f(r_s') \tag{18}$$

Carrying out the integral,

$$V(r_s) = \frac{4R_{ch}^2/r_s^2 + k_p^2 r_s^2 - e^{-2R_m^2/r_s^2} k_p^2 \left[2(R_{ch}^2 + R_m^2) + r_s^2\right]}{2k^2 R_{ch}^2} \tag{19}$$

Equation (17) is in the form of the total energy of a particle in a potential well. The first term is the effective kinetic energy and the second is the effective potential. The position of the particle, r_s, corresponds to the spot size of the beam. The effective potential for a leaky channel ($R_m \gtrsim R_{ch}$) is shown in Fig. 1. The first local minimum of V corresponds to the matched spot size r_M. If a beam is introduced into the channel with $r_s = r_M$ and $dr_s/dz = 0$, then the spot size remains constant. If a beam is introduced in some other condition its spot size will oscillate provided the energy of the particle, C, is below the potential barrier. If the energy of the particle exceeds the potential barrier it escapes the potential well and the beam expands indefinitely.

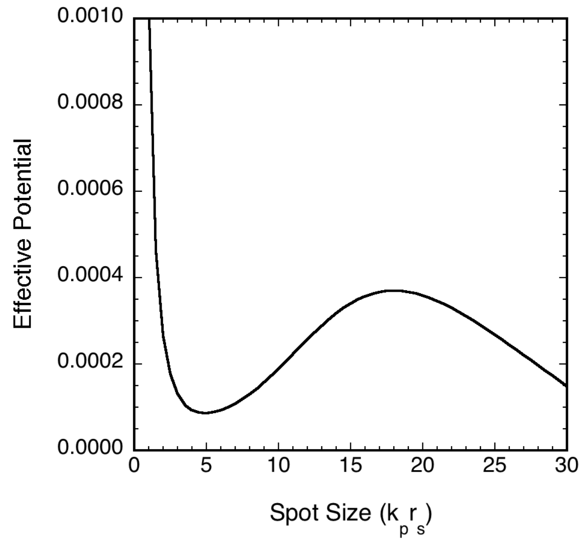

FIG. 1: Effective potential for $k/k_p = 44$, $k_pR_{ch} = 12$, and $k_pR_m = 20$.

The maximum energy of a confined particle is the height of the potential barrier, which is approximately $V(R_m)$. Therefore the condition for a confined beam is

$$\frac{1}{2}\left(\left.\frac{dr_s}{dz}\right|_{z=z_0}\right)^2 + V(r_{s0}) < V(R_m) \tag{20}$$

where z_0 is a reference point and $r_{s0} = r_s(z_0)$. Evaluating $V(R_m)$ gives

$$V(R_m) = \frac{k_p^2}{k^2}\left(-\frac{1}{e^2} + \frac{2}{k_p^2R_m^2} + \frac{R_m^2}{2R_{ch}^2} - \frac{3R_m^2}{2e^2R_{ch}^2}\right) \tag{21}$$

It often happens that $R_m^2/R_{ch}^2 \gg 1$ and $k_p^2R_m^2 \gg 1$ in which case

$$V(R_m) \approx \frac{k_p^2}{k^2}\frac{R_m^2}{R_{ch}^2}\left(\frac{1}{2} - \frac{3}{2e^2}\right) \approx 0.3\frac{k_p^2}{k^2}\frac{R_m^2}{R_{ch}^2} \tag{22}$$

Furthermore, if $r_{s0} \ll R_m$, then

$$V(r_{s0}) \approx \frac{2}{k^2r_{s0}^2} + \frac{1}{2}\frac{k_p^2}{k^2}\frac{r_{s0}^2}{R_{ch}^2} \tag{23}$$

Now consider the case where the beam is focused at the entrance of the plasma channel. Taking z_0 to be the position of the channel entrance, $dr_s/dz|_{z=z_0} = 0$, and the condition for a confined beam becomes

$$\frac{2}{k^2r_{s0}^2} + \frac{1}{2}\frac{k_p^2}{k^2}\frac{r_{s0}^2}{R_{ch}^2} < 0.3\frac{k_p^2}{k^2}\frac{R_m^2}{R_{ch}^2} \tag{24}$$

The first term on the left can be rewritten in terms of the f-number of the focusing giving

$$\frac{1}{8F_c^2} = \frac{k_p^2}{k^2}\left(0.3\frac{R_m^2}{R_{ch}^2} - \frac{1}{2}\frac{r_{s0}^2}{R_{ch}^2}\right) \tag{25}$$

Finally, noting that the second term on the right is usually small compared to the first, we obtain

$$F_c = 0.65 \frac{R_{ch}}{R_m} \frac{k}{k_p} \tag{26}$$

which is the same as the ray tracing estimate except for a constant factor of 1.3.

The SDE equations can also be used to obtain the matched spot size r_M and the frequency of the spot size oscillation. To obtain an analytical result, it is necessary to take $R_m \to \infty$ so that

$$V_\infty = V(r_s, R_m \to \infty) = \frac{4R_{ch}^2/r_s^2 + k_p^2 r_s^2}{2k^2 R_{ch}^2} \tag{27}$$

Solving $dV_\infty/dr_s = 0$ for r_s gives the matched spot size as

$$r_M^2 = \frac{2R_{ch}}{k_p} \tag{28}$$

The wavenumber of the spot size oscillation is

$$K^2 = \left.\frac{d^2V_\infty}{dr_s^2}\right|_{r_s=r_M} = \frac{4k_p^2}{k^2 R_{ch}^2} \tag{29}$$

This is just twice the frequency of the ray oscillation, as expected. In the case of finite R_m, the equation $dV/dr_s = 0$ has to be solved numerically.

V. EXAMPLE

As an example, consider a plasma channel with on-axis electron density of 10^{18} cm^{-3} and a laser pulse with wavelength 0.8 μm. Take $R_{ch} = 60$ μm and $R_m = 250$ μm. Then the critical f-number based on Eq. (26) is 6.9 and the period of the spot size oscillation based on Eq. (29) is $1660/k_p$ (about 9 mm). To evaluate the accuracy of these estimates, we solve the full equation of motion

$$\frac{d^2 r_s}{dz^2} + \frac{dV}{dr_s} = 0 \tag{30}$$

The results are plotted in Fig. 2. For f/6 focusing the beam escapes the channel, while for f/7 focusing the beam is guided. Hence, the critical f-number is between 6 and 7, which is consistent with the estimated value. It is easy to obtain a more accurate estimate of the critical f-number by numerically solving

$$V\left(\frac{4F_c}{k}\right) = V(R_m) \tag{31}$$

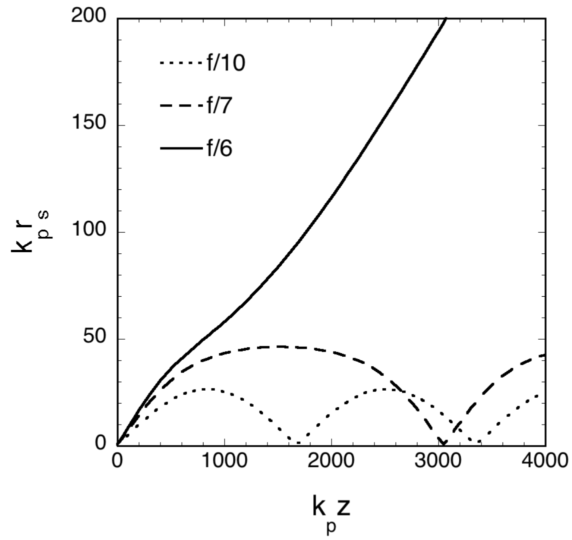

FIG. 2: Spot size as a function of propagation distance in the channel for f/6 focusing (solid curve), f/7 focusing (dashed curve) and f/10 focusing (dotted curve).

For the parameters considered here, this gives $F_c = 6.94$ as opposed to the analytical estimate of $F_c = 6.86$. The frequency of the spot size oscillation agrees with the estimated value in the case of f/10 focusing, but is lower in the case of f/7 focusing. Evidently the spot size oscillation slows down as the f-number approaches F_c.

VI. SUMMARY

Radiation propagating in a finite radius plasma channel can be lost if the focusing used to couple the radiation into the channel is too strong. The critical f-number for guiding is to a good approximation

$$F_c = 0.65 \frac{R_{ch}}{R_m} \frac{k}{k_p} \tag{32}$$

where R_{ch} is the density doubling radius and R_m is the maximum radius (beyond which the density stops increasing).

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